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# Fourier expansion of Arakawa lifting and central $L$ -values

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## Abstract

This note is a write-up of our talk at the RIMS-conference on automorphic forms held at the University of Tokyo in January 2010. Our results deal with the Fourier coefficients of Arakawa lifts and central values of some automorphic  $L$ -functions. In our previous paper [M-N-2] we provide an explicit formula for the Fourier coefficients in terms of toral integrals of some automorphic forms with respect to Hecke characters of imaginary quadratic fields. After studying explicit relations between the toral integrals and the central  $L$ -values, we explicitly determine the constant of proportionality relating the square norm of a Fourier coefficient of an Arakawa lift with the central  $L$ -value. In some case we can relate such square norm with the central value of some degree eight  $L$ -function of convolution type attached to the lift and the character. We also discuss the existence of strictly positive central values of the  $L$ -functions in our concern.

## 1 Reviews on Arakawa lifting and its Fourier expansion.

### 1.1

In this section we review our results in [M-N-2]. Let  $B$  be a definite quaternion algebra over  $\mathbb{Q}$  and denote by  $n$  its reduced norm. Let  $G = GSp(1, 1)$  be the  $\mathbb{Q}$ -algebraic group defined by

$$G_{\mathbb{Q}} := \{g \in M_2(B) \mid {}^t \bar{g} Q g = \nu(g) Q, \nu(g) \in \mathbb{Q}^{\times}\},$$

where  $Q := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . By  $Z_G$  we denote the center of  $G$ .

We put  $G_{\infty}^1 := \{g \in M_2(\mathbb{H}) \mid {}^t \bar{g} Q g = Q\}$ , where  $\mathbb{H} := B \otimes_{\mathbb{Q}} \mathbb{R}$  is the Hamilton quaternion algebra. Then

$$K_{\infty} := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a \pm b \in \mathbb{H}^1 \right\}$$

forms a maximal compact subgroup of  $G_{\infty}^1$ , where  $\mathbb{H}^1 := \{u \in \mathbb{H} \mid n(u) = 1\}$ . For a non-negative integer  $\kappa$  we let  $(\sigma'_{\kappa}, V_{\kappa})$  be the  $\kappa$ -th symmetric tensor representation of  $GL_2(\mathbb{C})$  and  $\sigma_{\kappa}$  the pull-back of  $\sigma'_{\kappa}$  to  $\mathbb{H}^{\times}$  via the standard embedding  $\mathbb{H} \subset M_2(\mathbb{C})$  (cf. [M-N-2, (1.4)]). This induces an irreducible representation  $(\tau_{\kappa}, V_{\kappa})$  of  $K$  by

$$\tau_{\kappa} \left( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right) := \sigma_{\kappa}(a + b), \quad \left( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in K_{\infty} \right).$$

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Let  $H$  and  $H'$  be  $\mathbb{Q}$ -algebraic groups defined by

$$H_{\mathbb{Q}} = GL_2(\mathbb{Q}), \quad H'_{\mathbb{Q}} := B^{\times}$$

respectively.

Fix a maximal order  $\mathfrak{O}$  of  $B$ . We denote by  $d_B$  the discriminant of  $B$  and fix a divisor  $D$  of  $d_B$ . For  $p|d_B$  let  $\mathfrak{P}_p$  be the maximal ideal of the  $p$ -adic completion  $\mathfrak{O}_p$  of  $\mathfrak{O}$  and let

$$L_p := \begin{cases} {}^t(\mathfrak{O}_p \oplus \mathfrak{O}_p) & (p \nmid d_B \text{ or } p|D), \\ {}^t(\mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}) & (p|d_B). \end{cases}$$

We put  $K_p := \{k \in G_p \mid kL_p = L_p\}$  for each finite prime  $p$  and  $K_f := \prod_{p < \infty} K_p$ . It is known that, up to  $G_p$ -conjugate, the two groups  $K_p$  for the two  $L_p$ 's exhaust maximal compact subgroups of  $G_p$ . For  $\kappa > 4$  we then introduce the space  $S_{\kappa}$  of  $V_{\kappa}$ -valued cusp forms  $F$  on  $G_{\mathbb{A}_{\mathbb{Q}}}$  satisfying the following:

1.  $F(z\gamma g k_f k_{\infty}) = \tau_{\kappa}(k_{\infty})^{-1}F(g)$  for  $(z, \gamma, g, k_f, k_{\infty}) \in Z_{G, \mathbb{A}_{\mathbb{Q}}} \times G_{\mathbb{Q}} \times G_{\mathbb{A}_{\mathbb{Q}}} \times K_f \times K_{\infty}$ ,
2. for each fixed  $g_f \in G_{\mathbb{A}_f}$ , the right translations of the coefficients of  $F|_{G_{\infty}^1}(g_f \cdot)$  by  $g_{\infty} \in G_{\infty}^1$  generate, as a  $(\mathfrak{g}, K_{\infty})$ -module, quaternionic discrete series with the minimal  $K_{\infty}$ -type  $\tau_{\kappa}$  (cf. [Gr-W]). Here  $\mathfrak{g}$  denotes the Lie algebra of  $G_{\infty}^1$ .

For a positive integer  $\kappa$  we let  $S_{\kappa}(D)$  be the space of elliptic cusp forms of weight  $\kappa$  with level  $D$  (cf. [M-N-2, §3.1]) and  $\mathcal{A}_{\kappa}$  be the space of automorphic forms of weight  $\sigma_{\kappa}$  with respect to  $\prod_{v < \infty} \mathfrak{O}_v^{\times}$  (cf. [M-N-2, §3.2]), where  $\mathfrak{O}_v^{\times}$  denotes the multiplicative group of  $\mathfrak{O}_v$ . Now we can review the definition of Arakawa lifting. By a metaplectic representation of  $GS(1, 1)_{\mathbb{A}_{\mathbb{Q}}} \times H_{\mathbb{A}_{\mathbb{Q}}} \times H'_{\mathbb{A}_{\mathbb{Q}}}$ , we define the  $\text{End}(V_{\kappa})$ -valued theta function  $\theta_{\kappa}(g, h, h')$  with some specified  $\text{End}(V_{\kappa})$ -valued Schwartz-Bruhat function on  $B_{\mathbb{A}}^{\oplus 2} \times \mathbb{A}_{\mathbb{Q}}^{\times}$  (cf. [M-N-1, §3]). Then, for  $\kappa > 4$ , we define the Arakawa lifting by

$$S_{\kappa}(D) \times \mathcal{A}_{\kappa} \ni (f, f') \mapsto \mathcal{L}(f, f')(g) \in S_{\kappa}$$

with

$$\mathcal{L}(f, f')(g) := \int_{\mathbb{R}_+^2 (H \times H')_{\mathbb{Q}} \backslash (H \times H')_{\mathbb{A}_{\mathbb{Q}}}} \overline{f(h)} \theta_{\kappa}(g, h, h') f'(h') dh dh'.$$

## 1.2

We now review the Fourier expansion of  $\mathcal{L}(f, f')$  described in [M-N-2, §1.3]. We let  $B^{-} := \{x \in B \mid \text{tr}(x) = 0\}$  and have

$$\mathcal{L}(f, f')(g) = \sum_{\xi \in B^{-} \setminus \{0\}} \mathcal{L}(f, f')_{\xi}(g),$$

where

$$\mathcal{L}(f, f')_{\xi}(g) := \int_{B^{-} \setminus B_{\mathbb{A}_{\mathbb{Q}}}^{-}} \mathcal{L}(f, f')\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\text{tr}(\xi x)) dx$$

with the standard additive character  $\psi$  on  $\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}$ . Here we normalize the measure  $dx$  so that the volume of  $B^- \backslash B_{\mathbb{A}_{\mathbb{Q}}}^-$  is one. For  $\xi \in B^- \setminus \{0\}$  we let  $E_{\xi} := \mathbb{Q}(\xi)$ , which is isomorphic to an imaginary quadratic field, and  $X_{\xi}$  be the set of unitary characters on  $\mathbb{A}_{\mathbb{Q}}^{\times} E_{\xi}^{\times} \backslash \mathbb{A}_{E_{\xi}}^{\times}$ . The Fourier expansion is then refined as follows:

$$\mathcal{L}(f, f')(g) = \sum_{\xi \in B^- \setminus \{0\}} \sum_{\chi \in X_{\xi}} \mathcal{L}(f, f')_{\xi}^{\chi}(g),$$

with

$$\mathcal{L}(f, f')_{\xi}^{\chi}(g) := \text{vol}(\mathbb{R}_+^{\times} \mathbb{A}_{\mathbb{Q}}^{\times} \backslash \mathbb{A}_{E_{\xi}}^{\times})^{-1} \int_{\mathbb{R}_+^{\times} E_{\xi}^{\times} \backslash \mathbb{A}_{E_{\xi}}^{\times}} \mathcal{L}(f, f')_{\xi}(s 1_2 \cdot g) \chi(s)^{-1} ds.$$

### 1.3

To review our explicit formula for  $\mathcal{L}(f, f')_{\xi}^{\chi}$ , we let  $(f, f') \in S_{\kappa}(D) \times \mathcal{A}_{\kappa}$  and assume the following two:

(1) The two forms  $f$  and  $f'$  are Hecke eigenforms and have the same eigenvalue for the “Atkin-Lehner involution”. More precisely, for each  $p|D$ , let  $\epsilon_p$  (resp.  $\epsilon'_p$ ) be the eigenvalue for the involutive action of  $\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$  (resp. a prime element  $\varpi_{B,p} \in B_p$ ) on  $f$  (resp.  $f'$ ). Then

$$\epsilon_p = \epsilon'_p.$$

Otherwise  $\mathcal{L}(f, f') \equiv 0$  (cf. [M-N-1, Remark 5.2 (ii)]).

(2) We assume that  $\xi \in B^- \setminus \{0\}$  is primitive. Namely, letting  $\mathfrak{a}_p := \begin{cases} \mathfrak{D}_p & (p \nmid d_B \text{ or } p|D) \\ \mathfrak{P}_p & (p|\frac{d_B}{D}) \end{cases}$

for each finite prime  $p$ ,

$$\xi \in \mathfrak{a}_p \setminus p\mathfrak{a}_p.$$

For the second assumption we note that, in general, a Fourier coefficient  $F_{\xi}$  of an automorphic form  $F$  on  $G_{\mathbb{A}_{\mathbb{Q}}}$  satisfies

$$F_{\xi}(g) = F_{\xi}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right) = F_{t\xi}(g) \quad (t \in \mathbb{Q}^{\times}).$$

We then see that the problem determining  $F_{\xi}$  is reduced to the case where  $\xi$  is primitive.

### 1.4

For each  $\xi \in B^- \setminus \{0\}$ ,  $d_{\xi}$  denotes the discriminant of  $E := E_{\xi}$ . We put

$$a := \begin{cases} 2\sqrt{-n(\xi)}\sqrt{d_{\xi}} & (d_{\xi} \text{ is odd}) \\ \sqrt{-n(\xi)}\sqrt{d_{\xi}} & (d_{\xi} \text{ is even}) \end{cases}, \quad b := \xi^2 - \frac{a^2}{4}.$$

With these  $a$  and  $b$  we define  $\iota_{\xi} : E^{\times} \hookrightarrow GL_2(\mathbb{Q})$  by

$$\iota_{\xi}(x + y\xi) = x \cdot 1_2 + y \cdot \begin{pmatrix} \frac{a}{2} & b \\ 1 & -\frac{a}{2} \end{pmatrix} \quad (x, y \in \mathbb{Q}).$$

Put  $\theta := r^{-1}(\xi - \frac{a}{2})$  with  $r = \frac{2\sqrt{n(\xi)}}{\sqrt{d_\xi}} \in \mathbb{Q}^\times$ . Then  $\{1, \theta\}$  forms a  $\mathbb{Z}$ -basis of the integer ring  $\mathcal{O}_E$  of  $E$ . We can rewrite  $\iota_\xi$  as

$$\iota_\xi(x + y\theta) = \begin{pmatrix} x & -rN_{E/\mathbb{Q}}(\theta)y \\ r^{-1}y & x + \text{Tr}_{E/\mathbb{Q}}(\theta)y \end{pmatrix} \quad (x, y \in \mathbb{Q}).$$

The completion  $E_\infty$  of  $E$  at  $\infty$  is identified with  $\mathbb{C}$  by

$$\delta_\xi : E_\infty \ni x + y\xi \mapsto x + y\sqrt{-n(\xi)} \in \mathbb{C} \quad (x, y \in \mathbb{R}).$$

For a Hecke character  $\chi = \prod_{v \leq \infty} \chi_v$  of  $\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times$ , we let  $w_\infty(\chi) \in \mathbb{Z}$  be such that

$$\chi_\infty(u) = (\delta_\xi(u)/|\delta_\xi(u)|)^{w_\infty(\chi)} \quad (u \in E_\infty).$$

Furthermore, for each prime  $v = p < \infty$ , we let  $p^{i_p(\chi)}$  be the conductor of  $\chi$  at  $p$  and

$$\mu_p := \frac{\text{ord}_p(2\xi)^2 - \text{ord}_p(d_\xi)}{2},$$

which coincides with  $\text{ord}_p(r)$ . We now state the following (cf. [M-N-2, Theorem 5.1.1]):

**Proposition 1.1.**  $\mathcal{L}(f, f')_\xi^\chi \equiv 0$  unless  $i_p(\chi) = 0$  for any  $p|d_B$  and  $w_\infty(\chi) = -\kappa$ .

## 1.5

In what follows, we assume that  $\chi$  satisfies the assumption in the proposition above. We need further notations to recall our formula for  $\mathcal{L}(f, f')_\xi^\chi$ .

We define  $\gamma_0 = (\gamma_{0,p})_{p \leq \infty} \in H_{\mathbb{A}_\mathbb{Q}}$  and  $\gamma'_0 = (\gamma'_{0,p})_{p < \infty} \in H'_{\mathbb{A}_f}$  as follows:

$$\gamma_{0,p} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p + i_p(\chi)} \end{pmatrix} & (p \nmid D), \\ 1_2 & (p|D \text{ and } p \text{ is inert in } E), \\ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (p|D \text{ and } p \text{ ramifies in } E), \\ n_H(a/2)d_H(N(\xi)^{1/4}) & (p = \infty), \end{cases}$$

$$\gamma'_{0,p} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p + i_p(\chi)} \end{pmatrix} & (p \nmid d_B), \\ \varpi_{B,p}^{-1} & (p|d_B). \end{cases}$$

Here recall that  $\varpi_{B,p}$  denotes a prime element of  $B_p$  (cf. §1.3).

In addition, we introduce the following local constants:

$$C_p(f, \xi, \chi) := \begin{cases} p^{2\mu_p - i_p(\chi)}(1 - \delta(i_p(\chi) > 0)e_p(E)p^{-1}) & (p \nmid d_B), \\ 1 & (p|\frac{d_B}{D}), \\ 2\epsilon_p & (p|D \text{ and } p \text{ is inert in } E), \\ (p+1)^{-1} & (p|D \text{ and } p \text{ ramifies in } E), \end{cases}$$

where

$$e_p(E) = \begin{cases} -1 & (p \text{ is inert in } E), \\ 0 & (p \text{ ramifies in } E), \\ 1 & (p \text{ splits in } E). \end{cases}$$

For  $(f, f') \in S_\kappa(D) \times \mathcal{A}_\kappa$  we introduce their toral integrals with respect to  $\chi$  (cf. [Wa]).

$$P_\chi(f; h) := \int_{\mathbb{R}_+^\times E_\xi^\times \backslash \mathbb{A}_{E_\xi}^\times} f(\iota_\xi(s)h) \chi(s)^{-1} ds, \quad P_\chi(f'; h') := \int_{\mathbb{R}_+^\times E_\xi^\times \backslash \mathbb{A}_{E_\xi}^\times} f(sh') \chi(s)^{-1} ds,$$

where  $(h, h') \in GL_2(\mathbb{A}_\mathbb{Q}) \times B_{\mathbb{A}_\mathbb{Q}}^\times$ . As in [M-N-2, §2.4] we normalize the measure  $ds$  of  $\mathbb{A}_E^\times$  so that

$$\text{vol}(\mathcal{O}_{E,p}^\times) = 1 \text{ for any } p < \infty, \quad \text{vol}(E_\infty^1) = 1.$$

We denote by  $\mathbf{h}(E)$  and  $w(E)$  the class number of  $E$  and the number of the roots of unity in  $E$  respectively. Then we are able to state our formula for  $\mathcal{L}(f, f')_\xi^\chi$  (cf. [M-N-2, Theorem 5.2.1]).

**Theorem 1.2.** *Let  $(f, f')$  be Hecke eigenforms and  $\xi \in B^- \setminus \{0\}$  be primitive. Suppose that  $\chi$  satisfies the assumption in Proposition 1.1 and that (1) and (2) in §1.3 hold. We then have the following formula:*

$$\begin{aligned} & \mathcal{L}(f, f')_\xi^\chi(g_{0,f} d_G(\sqrt{\eta_\infty})) \\ &= 2^{\kappa-1} n(\xi)^{\kappa/4} \frac{w(E)}{\mathbf{h}(E)} \cdot \left( \prod_{p < \infty} C_p(f, \xi, \chi) \right) \eta_\infty^{\kappa/2+1} \exp(-4\pi\sqrt{n(\xi)\eta_\infty}) \overline{P_\chi(f; \gamma_0)} P_\chi(f'; \gamma'_0). \end{aligned}$$

Here  $\eta_\infty \in \mathbb{R}_+^\times$  and  $g_{0,f} = (g_{0,p})_{p < \infty} \in G_{\mathbb{A}_f}$  is given by

$$g_{0,p} := \begin{cases} \text{diag}(p^{i_p(\chi)-\mu_p}, p^{2(i_p(\chi)-\mu_p)}, 1, p^{i_p(\chi)-\mu_p}) & (p \nmid d_B), \\ 1_2 & (p \mid d_B) \end{cases}$$

**Remark 1.3.** According to Sugano [Su, Theorem 2-1], the Fourier coefficient  $\mathcal{L}(f, f')_\xi^\chi$  is determined by the evaluation at  $g_{0,f} \begin{pmatrix} \sqrt{\eta_\infty} & 0 \\ 0 & \sqrt{\eta_\infty}^{-1} \end{pmatrix}$ .

## 2 Relation with central $L$ -values.

### 2.1

Let  $(f, f')$  be Hecke eigenforms and assume that  $f$  is a primitive form (for the definition, see [Mi, §4.6]). Let  $\pi(f)$  (resp.  $\pi(f')$ ) be the irreducible automorphic representation generated by  $f$  (resp.  $f'$ ), and let  $\pi(\text{JL}(f'))$  be the irreducible automorphic representation generated by the Jacquet-Langlands lift  $\text{JL}(f')$  of  $f'$ . It is known that  $\pi(f)$  and  $\pi(\text{JL}(f'))$  (resp.  $\pi(f')$ ) decompose into restricted tensor products over  $v \leq \infty$  of irreducible admissible representations of  $GL_2(\mathbb{Q}_v)$  (resp.  $B_v^\times$ ). By  $\pi_v$ ,  $\pi'_v$  and  $\pi''_v$  we denote the  $v$ -component of  $\pi(f)$ ,

$\pi(\text{JL}(f'))$  and  $\pi(f')$  respectively. According to such decomposition of  $\pi(f)$  and  $\pi(f')$ ,  $f$  and  $f'$  admit decompositions into pure tensor products

$$\rho(f) = \prod_{v \leq \infty} f_v, \quad \rho'(f') = \prod_{v \leq \infty} f'_v,$$

where we fix an isomorphism  $\rho$  (resp.  $\rho'$ ) between  $\pi(f)$  and  $\otimes'_{v \leq \infty} \pi_v$  (resp.  $\pi(f')$  and  $\otimes'_{v \leq \infty} \pi'_v$ ). We denote by  $\Pi$  (resp.  $\Pi'$ ) the quadratic base change of  $\pi(f)$  (resp.  $\pi(\text{JL}(f'))$ ) to  $GL_2(\mathbb{A}_E)$ . These  $\Pi$  and  $\Pi'$  also decompose into the restricted tensor products  $\otimes'_{v \leq \infty} \Pi_v$  and  $\otimes'_{v \leq \infty} \Pi'_v$  respectively, where each  $\Pi_v$  or  $\Pi'_v$  is a local base change lift of  $\pi_v$  or  $\pi'_v$  at every place  $v$  respectively.

## 2.2 Review on the adjoint $L$ -functions and the $L$ -functions of base change lifts for $GL_2$ .

Let  $L(\pi, s)$  be the standard  $L$ -function for an automorphic representation  $\pi$  of  $GL_2(\mathbb{A}_{\mathbb{Q}})$  in the sense of Jacquet-Langlands [J-L]. We denote by  $L(\Pi, \chi^{-1}, s)$  (resp.  $L(\Pi', \chi^{-1}, s)$ ) the  $L$ -function of  $\Pi$  (resp.  $\Pi'$ ) with  $\chi^{-1}$ -twist, and let  $L(\pi(f), \text{Ad}, s)$  (resp.  $L(\pi(\text{JL}(f')), \text{Ad}, s)$ ) be the adjoint  $L$ -function of  $\pi(f)$  (resp.  $\pi(\text{JL}(f'))$ ).

We describe the local factors of  $L(\Pi, \chi^{-1}, s)$  and  $L(\Pi', \chi^{-1}, s)$  (resp.  $L(\pi(f), \text{Ad}, s)$  and  $L(\pi(\text{JL}(f')), \text{Ad}, s)$ ), following Jacquet [J] (resp. Gelbart-Jacquet [G-J]). We note that  $\pi_p$  (resp.  $\pi'_p \simeq \pi''_p$ ) is a unitary unramified principal series representation for each finite prime  $p$  (resp.  $p \nmid d_B$ ). This is due to the Ramanujan conjecture for holomorphic cusp forms on  $GL_2$ . In addition, we remark that, for  $p \mid d_B$ ,  $\pi''_p$  is written as

$$B_p^\times \ni b \mapsto \delta_p \cdot n(b) \in \{\pm 1\},$$

with a character  $\delta_p$  of  $\mathbb{Q}_p^\times$  of order two. For  $p \nmid d_B$ ,  $\pi'_p$  is thus the special representation of  $GL_2(\mathbb{Q}_p)$  given by the irreducible subquotient of the induced representation  $\text{Ind}(\delta_p \cdot |*|_p^{\frac{1}{2}}, \delta_p \cdot |*|_p^{-\frac{1}{2}})$  (cf. [J-L, Theorem 4.2 (iii)]), where  $|*|_p$  is the  $p$ -adic absolute value. We furthermore note that, when  $p$  is inert or ramified in  $E$  and  $\chi_p$  is unramified,  $\chi_p$  can be written as

$$\chi_p = \omega_p \cdot n_{E_p/\mathbb{Q}_p}$$

with a character  $\omega_p$  of  $\mathbb{Q}_p^\times$  of order two and the norm  $n_{E_p/\mathbb{Q}_p}$  of  $E_p$ . In fact,  $\omega_p \equiv 1$  when  $p$  is inert. In addition, at the Archimedean place,  $\pi_\infty$  and  $\pi'_\infty$  are the discrete series representations with weight  $\kappa$  and  $\kappa + 2$  respectively (see [Sh, §6]).

We let  $\pi_{\text{ur}}$  be a unitary unramified principal series representation of  $GL_2(\mathbb{Q}_p)$  with Satake parameter  $(\alpha_p, \alpha_p^{-1})$  and the trivial central character, and let  $\pi_{\text{sp}}$  be the special representation of  $GL_2(\mathbb{Q}_p)$  corresponding to  $\pi''_p$ . We denote by  $\Pi_{\text{ur}}$  (resp.  $\Pi_{\text{sp}}$ ) the base change lift of  $\pi_{\text{ur}}$  (resp.  $\pi_{\text{sp}}$ ) to  $GL_2(E_p)$ .

We first deal with the local  $L$ -functions of  $\pi_{\text{ur}}$  and  $\Pi_{\text{ur}}$ .

**Lemma 2.1.** (1)

$$L_p(\pi_{\text{ur}}, s) = (1 - \alpha_p p^{-s})^{-1} (1 - \alpha_p^{-1} p^{-1})^{-1}.$$

(2)

$$L_p(\pi_{\text{ur}}, \text{Ad}, s) = (1 - p^{-s})^{-1} (1 - \alpha_p^2 p^{-s})^{-1} (1 - \alpha_p^{-2} p^{-s})^{-1}.$$

(3)

$$L_p(\Pi_{\text{ur}}, \chi_p^{-1}, s) = \begin{cases} \prod_{i=1,2} (1 - \alpha_p \chi_p(\varpi_{p,i})^{-1} p^{-s})^{-1} (1 - \alpha_p^{-1} \chi_p(\varpi_{p,i})^{-1} p^{-s})^{-1} & (p:\text{split}, i_p(\chi) = 0), \\ (1 - \alpha_p^2 p^{-2s})^{-1} (1 - \alpha_p^{-2} p^{-2s})^{-1} & (p:\text{inert}, i_p(\chi) = 0), \\ (1 - \alpha_p \chi_p(\varpi_p)^{-1} p^{-s})^{-1} (1 - \alpha_p^{-1} \chi_p(\varpi_p)^{-1} p^{-s})^{-1} & (p:\text{ramified}, i_p(\chi) = 0), \\ 1 & (i_p(\chi) > 0), \end{cases}$$

where  $\varpi_{p,i} \in E_p$  with  $i = 1, 2$  (resp.  $\varpi_p \in E_p$ ) denote two distinct prime elements dividing  $p$  (a prime element dividing  $p$ ) when  $p$  is split (resp.  $p$  is ramified).

We next deal with the case of  $\pi_{\text{sp}}$  and  $\Pi_{\text{sp}}$ .

**Lemma 2.2.** *We have*

$$L_p(\pi_{\text{sp}}, s) = (1 - \delta_p(p) p^{-(s+\frac{1}{2})})^{-1}, \quad (2.1)$$

$$L_p(\pi_{\text{sp}}, \text{Ad}, s) = (1 - p^{-(s+1)})^{-1}, \quad (2.2)$$

$$L_p(\Pi_{\text{sp}}, \chi_p^{-1}, s) = \begin{cases} (1 - p^{-(2s+1)})^{-1} & (p:\text{inert}), \\ (1 - \delta_p(p) \omega_p(p) p^{-(s+\frac{1}{2})})^{-1} & (p:\text{ramified}). \end{cases} \quad (2.3)$$

For a positive integer  $k \geq 2$  we let  $\pi_k$  be the discrete series representation with weight  $k$  and  $\Pi_k$  denote its base change. We give their Archimedean local  $L$ -functions as follows.

**Lemma 2.3.** *We have*

$$L_\infty(\pi_k, s) = \Gamma_{\mathbb{C}}(s + \frac{k-1}{2}), \quad (2.4)$$

$$L_\infty(\pi_k, \text{Ad}, s) = \Gamma_{\mathbb{R}}(s+1) \Gamma_{\mathbb{C}}(s+k-1), \quad (2.5)$$

$$L_\infty(\Pi_k, \chi_l, s) = \begin{cases} \Gamma_{\mathbb{C}}(s + \frac{k-1}{2} + |l|) \Gamma_{\mathbb{C}}(s - \frac{k-1}{2} + |l|) & (|l| \geq \frac{\kappa-1}{2}) \\ \Gamma_{\mathbb{C}}(s + \frac{k-1}{2} + |l|) \Gamma_{\mathbb{C}}(s + \frac{k-1}{2} - |l|) & (|l| \leq \frac{\kappa-1}{2}) \end{cases}, \quad (2.6)$$

where  $l \in \frac{1}{2}\mathbb{Z}$ .

### 2.3 Relation between $P_\chi(f; \gamma_0)$ and $L(\Pi, \chi^{-1}, \frac{1}{2})$ .

By  $\eta$  we denote the quadratic character attached to the quadratic extension  $E/\mathbb{Q}$ . We let  $L(\eta, s)$  be the  $L$ -function defined by  $\eta$  and  $L_v(\eta_v, s)$  the local factor of  $L(\eta, s)$  at a place  $v$ . Let  $\mathcal{T}_p^\pm f = \lambda_p^\pm f$  for  $p|D$  (for the definition of  $\mathcal{T}_p^\pm$ , see [Mu, 2.4]) and let

$$\begin{aligned} S_1 &:= \{p < \infty \mid p|D, p \text{ is inert in } E\}, \\ S_2^\pm &:= \{p < \infty \mid p|D, p \text{ ramifies in } E, \chi_p(\varpi_p) \lambda_p^\pm = \pm 1\}, \end{aligned}$$



where  $\varpi_p$  denotes a prime element of  $E_p$  (cf. Lemma 2.1). Note that  $S_1 \cup S_2^+ \cup S_2^-$  coincides with  $\delta(D) := \{p < \infty \mid p \mid D\}$ . We furthermore put  $A(\chi) := \prod_{p < \infty} p^{i_p(\chi)}$ .

Let us introduce the Petersson norm

$$\langle f, f \rangle := \int_{Z(\mathbf{A}_Q)GL_2(\mathbb{Q}) \backslash GL_2(\mathbf{A}_Q)} |f(g)|^2 dg$$

of  $f$ , where  $Z$  denotes the center of  $GL_2$ . Keeping the assumption on  $\chi$  in Proposition 1.1 we can quote [Mu, Theorem 1.1] with a modification.

**Proposition 2.4.** *We have*

$$\frac{|P_\chi(f; \gamma_0)|^2}{\langle f, f \rangle} = C(f, \chi) L(\Pi, \chi^{-1}, \frac{1}{2}),$$

with

$$C(f, \chi) := \begin{cases} \frac{2^{|\delta(D)|} |d_\xi| \prod_{p \mid A(\chi)} L_p(\eta_p, 1)^2}{4D^{\frac{3}{2}} A(\chi) L(\pi(f), \text{Ad}, 1)} & (S_1 = S_2^+ = \emptyset), \\ 0 & (\text{otherwise}). \end{cases}$$

## 2.4 Relation between $P_\chi(f'; \gamma'_0)$ and $L(\Pi', \chi^{-1}, \frac{1}{2})$ .

Next we provide an explicit relation between  $\|P_\chi(f'; \gamma'_0)\|^2$  and the central L-value  $L(\Pi', \chi^{-1}, \frac{1}{2})$ , which we prove in §3. To write down the relation we need several notations. We denote by  $r_p$

the ramification index of  $p$  for the quadratic extension  $E/\mathbb{Q}$ , i.e.  $r_p := \begin{cases} 1 & (p: \text{non-ramified}) \\ 2 & (p: \text{ramified}) \end{cases}$ .

We set

$$\langle f', f' \rangle := \int_{Z'(\mathbf{A}_Q) B^\times \backslash B_{\mathbf{A}_Q}^\times} (f'(b), f'(b))_\kappa db,$$

where  $Z'$  denotes the center of  $B^\times$  and  $(*, *)_\kappa$  stands for an inner product of  $V_\kappa$ . By  $\langle *, * \rangle_\infty$  we denote an inner product of  $\pi''_\infty$ . As this inner product we can take

$$\langle h_\infty, h'_\infty \rangle_\infty := \int_{\mathbb{H}^1} (h_\infty(u), h'_\infty(u))_\kappa du \quad (h_\infty, h'_\infty \in \pi''_\infty \simeq V_\kappa),$$

where  $du$  denotes the invariant measure of  $\mathbb{H}^1$  normalized so that  $\text{vol}(\mathbb{H}^1) = 1$ . Let  $v_{\kappa, \xi}$  be a highest weight vector of  $V_\kappa$  with respect to  $\sigma_\kappa(\mathbb{R}(\xi)^\times)$ -action, and let  $v_{\kappa, \xi}^* \in V_\kappa$  be the dual of  $v_{\kappa, \xi}$  with respect to  $\langle *, * \rangle_\infty$ . We set  $f'_{\infty, \kappa}(b_\infty) := \langle f'_\infty(b_\infty), v_{\kappa, \xi}^* \rangle_\infty v_{\kappa, \xi}$  for  $b_\infty \in B_\infty^\times$ . The notation  $\zeta(s) = \prod_{v \leq \infty} \zeta_v(s)$  stands for the Riemann zeta function with a local factor  $\zeta_v(s)$  at each place  $v$ . With the assumption on  $\chi$  in Proposition 1.1 we then have the following:

**Proposition 2.5.**

$$\frac{\|P_\chi(f'; \gamma'_0)\|^2}{\langle f', f' \rangle} = \begin{cases} C(f', \chi) L(\Pi', \chi^{-1}, \frac{1}{2}) & (\pi_p''|_{E_p^\times} = \chi_p \text{ when } p \text{ divides } d_B \text{ and is ramified in } E), \\ 0 & (\text{otherwise}), \end{cases}$$

where

$$C(f', \chi) := \frac{\sqrt{|d_\xi|}(\kappa + 1)}{2^5 \pi A(\chi)} \prod_{p \mid A(\chi)} L_p(\eta_p, 1)^2 \cdot \prod_{p \mid d_B} r_p(1-p^{-1}) \cdot \frac{\langle f'_{\infty, \kappa}, f'_{\infty, \kappa} \rangle_\infty}{\langle f'_\infty, f'_\infty \rangle_\infty} \cdot \frac{\zeta(2)}{L(\pi(\text{JL}(f')), \text{Ad}, 1)}.$$

## 2.5 Main result (first form)

Theorem 1.2, Proposition 2.4 and Proposition 2.5 imply the theorem as follows:

**Theorem 2.6.** *Under the assumption in Theorem 1.2 we have*

$$\frac{||\mathcal{L}(f, f')_{\xi}^{\chi}(g_0)||^2}{\langle f, f \rangle \langle f', f' \rangle} = C(f, f', \xi, \chi) L(\Pi, \chi^{-1}, \frac{1}{2}) L(\Pi', \chi^{-1}, \frac{1}{2}),$$

where, if  $\pi_p''|_{E_p^{\times}} = \chi_p$  for  $p|d_B$  ramified in  $E$  and  $S_1 = S_2^+ = \emptyset$ ,  $C(f, f', \xi, \chi) =$

$$\frac{2^{2\kappa+|\delta(D)|-9}(\kappa+1)n(\xi)^{\frac{\kappa}{2}}|d_{\xi}|^{\frac{3}{2}}w(E)^2}{\pi h(E)^2 A(\chi)^2 D^{\frac{3}{2}}} \prod_{p|A(\chi)} L_p(\eta_p, 1)^4 \prod_{p|d_B} r_p(1-p^{-1}) \prod_{p<\infty} C_p(f, \xi, \chi)^2 \\ \times e^{-8\pi\sqrt{n(\xi)}} \cdot \frac{\langle f'_{\infty, \kappa}, f'_{\infty, \kappa} \rangle_{\infty}}{\langle f'_{\infty}, f'_{\infty} \rangle_{\infty}} \cdot \frac{\zeta(2)}{L(\pi(f), \text{Ad}, 1) L(\pi(\text{JL}(f')), \text{Ad}, 1)},$$

and  $C(f, f', \xi, \chi) = 0$  otherwise.

## 2.6 Main result (second form)

We introduce the degree eight  $L$ -function attached to  $\mathcal{L}(f, f')$  and  $\chi$ , and relate its central value to the square norm  $||\mathcal{L}(f, f')_{\xi}^{\chi}(g_0)||^2$  when  $D = 1$ .

We now recall that  $\mathcal{L}(f, f')$  belongs to  $\mathcal{S}_{\kappa}$  (cf. §1.1). Before introducing the degree eight  $L$ -function, we define the spinor  $L$ -function for a Hecke eigenform  $F \in \mathcal{S}_{\kappa}$ . In [M-N-1, §5.1] we introduced three Hecke operator  $T_p^i$  with  $0 \leq i \leq 2$  for  $p \nmid d_B$ . Let  $\Lambda_p^i$  be the Hecke eigenvalue of  $T_p^i$  for  $F$  with  $0 \leq i \leq 2$ . For  $p \nmid d_B$  we put

$$Q_{F,p}(t) := 1 - p^{-\frac{3}{2}} \Lambda_p^1 t + p^{-2} (\Lambda_p^2 + p^2 + 1) t^2 - p^{-\frac{3}{2}} \Lambda_p^1 t^3 + t^4.$$

For this we note that  $Q_{F,p}(p^{-s})^{-1}$  coincides with the local spinor  $L$ -function for an unramified principal series of the group of  $\mathbb{Q}_p$ -rational points for the split symplectic  $\mathbb{Q}$ -group  $GS(2)$  of degree two with similitudes. Here recall that  $GS(2)$  is defined by

$$GS(2)_{\mathbb{Q}} := \left\{ g \in GL(4)_{\mathbb{Q}} \mid {}^t g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}, \nu(g) \in \mathbb{Q}^{\times} \right\}.$$

On the other hand, in [M-N-1, §5.2], we introduced two Hecke operators  $T_p^i$  with  $0 \leq i \leq 1$  for  $p|d_B$ . Let  $\Lambda_p^i$  be the Hecke eigenvalue of  $T_p^i$  for  $F$  with  $0 \leq i \leq 1$ . For  $p|d_B$  we put

$$Q_{F,p}(t) := (1 - p^{-\frac{3}{2}} (\Lambda_p^1 - (p^{A_p} - 1) \Lambda_p^0) t + t^2) (1 - \Lambda_p^0 p^{-\frac{1}{2}} t),$$

where  $A_p := \begin{cases} 1 & (p \nmid D) \\ 2 & (p|D) \end{cases}$ . This is due to Sugano [Su, (3.4)].

We define the spinor  $L$ -function  $L(F, \text{spin}, s)$  of  $F$  by

$$L(F, \text{spin}, s) := \prod_{v \leq \infty} L_v(F, \text{spin}, s),$$

where

$$L_v(F, \text{spin}, s) := \begin{cases} Q_{F,p}(p^{-s})^{-1} & (v = p < \infty), \\ \Gamma_{\mathbb{C}}(s + \frac{\kappa-1}{2})\Gamma_{\mathbb{C}}(s + \frac{\kappa+1}{2}) & (v = \infty). \end{cases}$$

This is a modification of the definition in [M-N-1, §5.3]. We can then reformulate [M-N-1, Corollary 5.3] when  $D = 1$ .

**Proposition 2.7.** *The spinor  $L$ -function for  $\mathcal{L}(f, f')$  decomposes into*

$$L(\mathcal{L}(f, f'), \text{spin}, s) = L(\pi(f), s)L(\pi(\text{JL}(f')), s).$$

Of course, when  $D = 1$ , we see that  $L(\mathcal{L}(f, f'), \text{spin}, s)$  has the meromorphic continuation and satisfies the functional equation between  $s$  and  $1 - s$  since so do  $L(\pi(f), s)$  and  $L(\pi(\text{JL}(f')), s)$ .

We now introduce the  $L$ -function

$$L(F, \chi^{-1}, s) := \prod_{v \leq \infty} L_v(F, \chi^{-1}, s)$$

of degree eight for a Hecke eigenform  $F \in \mathcal{S}_{\kappa}$  and  $\chi$ . Here the local factors  $L_v(F, \chi^{-1}, s)$  are given as

$$L_v(F, \chi^{-1}, s) := \begin{cases} Q_{F,p}(\alpha_p^{\chi} p^{-s})^{-1} Q_{F,p}(\beta_p^{\chi} p^{-s})^{-1} & (\chi \text{ is unramified at } v = p < \infty) \\ 1 & (\chi \text{ is ramified at } v = p < \infty), \\ \Gamma_{\mathbb{C}}(s + \kappa - \frac{1}{2})\Gamma_{\mathbb{C}}(s + \frac{1}{2})\Gamma_{\mathbb{C}}(s + \kappa + \frac{1}{2})\Gamma_{\mathbb{C}}(s + \frac{1}{2}) & (v = \infty), \end{cases}$$

where

$$(\alpha_v^{\chi}, \beta_v^{\chi}) := \begin{cases} (\chi_p(\varpi_{p,1})^{-1}, \chi_p(\varpi_{p,2})^{-1}) & (v = p: \text{split}), \\ (\chi_p(p)^{-1}, -\chi_p(p)^{-1}) = (1, -1) & (v = p: \text{inert}), \\ (\chi_p(\varpi_p)^{-1}, 0) & (v = p: \text{ramified}) \end{cases}$$

with prime elements  $\varpi_{p,1}$ ,  $\varpi_{p,2}$  and  $\varpi_p$  introduced in Lemma 2.1.

**Proposition 2.8.** *Let  $D = 1$ . We have*

$$L(\mathcal{L}(f, f'), \chi^{-1}, s) = L(\Pi, \chi^{-1}, s)L(\Pi', \chi^{-1}, s).$$

*This has meromorphic continuation to  $\mathbb{C}$  and satisfies the functional equation*

$$L(\mathcal{L}(f, f'), \chi^{-1}, s) = \epsilon(\Pi, \chi^{-1})\epsilon(\Pi', \chi^{-1})L(\mathcal{L}(f, f'), \chi^{-1}, 1 - s),$$

*where  $\epsilon(\Pi, \chi^{-1})$  (resp.  $\epsilon(\Pi', \chi^{-1})$ ) denotes the  $\epsilon$ -factor of  $L(\Pi, \chi^{-1}, s)$  (resp.  $L(\Pi', \chi^{-1}, s)$ ).*

In view of Proposition 2.4, Proposition 2.5 and Proposition 2.8, the central value  $L(\mathcal{L}(f, f'), \chi^{-1}, \frac{1}{2})$  is meaningful when  $D = 1$ . We are now able to reformulate Theorem 2.6 as follows:

**Theorem 2.9.** *Let the assumption be as in Theorem 1.2 and let  $D = 1$ . We have*

$$\frac{\|\mathcal{L}(f, f')_{\xi}^{\chi}(g_0)\|^2}{\langle f, f \rangle \langle f', f' \rangle} = C(f, f', \xi, \chi)L(\mathcal{L}(f, f'), \chi^{-1}, \frac{1}{2}).$$

## 2.7 Main result (third form)

Note that the group  $G = GSp(1, 1)$  is an inner form of the symplectic  $\mathbb{Q}$ -group  $GSp(2)$  of degree two with similitudes. As is well-known, the Langlands principle of functoriality [Lg-1] suggests that the  $L$ -function for a Hecke eigenform  $F \in \mathcal{S}_\kappa$  should be an  $L$ -function for some automorphic form (or automorphic representation) of  $GSp(2)_{\mathbb{A}_\mathbb{Q}}$ .

T. Okazaki [O] has recently constructed cusp forms on  $GSp(2)_{\mathbb{A}_\mathbb{Q}}$ , which contain forms whose spinor  $L$ -functions coincide with those of  $\mathcal{L}(f, f')$ 's. More precisely his construction uses a theta lifting from  $GL(2)_{\mathbb{A}_\mathbb{Q}} \times GL(2)_{\mathbb{A}_\mathbb{Q}}$  to  $GSp(2)_{\mathbb{A}_\mathbb{Q}}$  (or from  $GO(2, 2)_{\mathbb{A}_\mathbb{Q}}$  to  $GSp(2)_{\mathbb{A}_\mathbb{Q}}$ ) and follows the formulation of Harris-Kudla [H-K]. He specifies an appropriate Schwartz-Bruhat function on  $M_2(\mathbb{A}_\mathbb{Q})^{\oplus 2}$  to construct the theta kernel for the lifting. We denote this lifting by  $\mathcal{L}_{GSp(2)}$ . His result is then stated as follows:

**Theorem 2.10** (Okazaki). *For two non-zero primitive cusp forms  $(f_1, f_2) \in S_{\kappa_1}(N_1) \times S_{\kappa_2}(N_2)$ ,  $\mathcal{L}_{GSp(2)}(f_1, f_2)$  is a non-zero generic cusp form on  $GSp(2)_{\mathbb{A}_\mathbb{Q}}$  with the following properties:*

1. *it is a paramodular form of level  $N_1 N_2$ , namely, at a prime  $p | N := N_1 N_2$ , it is right invariant by a paramodular group*

$$\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & N^{-1}\mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & N\mathbb{Z}_p & \mathbb{Z}_p & N\mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cap GSp(2)_{\mathbb{Q}_p},$$

2. *at the Archimedean place, the highest weight of  $\mathcal{L}_{GSp(2)}(f_1, f_2)$  is*

$$\left( \frac{\kappa_1 + \kappa_2}{2}, -\frac{|\kappa_1 - \kappa_2|}{2} \right).$$

*The global spinor  $L$ -function of  $\mathcal{L}_{GSp(2)}(f_1, f_2)$  (in the sense of Novodvorsky [No]) decomposes into*

$$L(\pi(f_1), s) L(\pi(f_2), s).$$

Also for  $\mathcal{L}_{GSp(2)}(f_1, f_2)$  and a Hecke character  $\chi$  we can similarly define the degree eight  $L$ -function  $L(\mathcal{L}_{GSp(2)}(f_1, f_2), \chi^{-1}, s)$ . For a Hecke eigenform  $f' \in \mathcal{A}_\kappa$  we have put  $JL(f')$  to be the Jacquet-Langlands lift of  $f'$  (cf. §2.1) and note that  $JL(f')$  is primitive (cf. [E-1], [E-2], [Sh, §6]). From the theorem just mentioned and Proposition 2.8 we deduce the following:

**Proposition 2.11.** *Let  $(f, f') \in S_\kappa(1) \times \mathcal{A}_\kappa$  be Hecke eigenforms with  $\mathcal{L}(f, f') \neq 0$ . Then we have*

$$\begin{aligned} L(\mathcal{L}(f, f'), \text{spin}, s) &= L(\mathcal{L}_{GSp(2)}(f, JL(f')), \text{spin}, s), \\ L(\mathcal{L}(f, f'), \chi^{-1}, s) &= L(\mathcal{L}_{GSp(2)}(f, JL(f')), \chi^{-1}, s). \end{aligned}$$

We thus know that Theorem 2.9 is restated as follows:

**Theorem 2.12.** *Let the assumption be as in Theorem 1.2 and let  $D = 1$ . We have*

$$\frac{||\mathcal{L}(f, f')_{\xi}^{\chi}(g_0)||^2}{\langle f, f \rangle \langle f', f' \rangle} = C(f, f', \xi, \chi) L(\mathcal{L}_{GSp(2)}(f, \text{JL}(f')), \chi^{-1}, \frac{1}{2}).$$

**Remark 2.13.** (1) This theorem is compatible with the conjecture of Furusawa-Martin [F-M] and Furusawa-Shalika [F-S], which are inspired by Böcherer [B].

(2) The example of the functorial correspondence above is essentially predicted by T. Ibukiyama in his study of automorphic forms on the compact inner form of  $GSp(2)$  (cf. [I]). We can expect that automorphic forms in  $S_{\kappa}$  (cf. §1.1) with  $D = 1$  should functorially correspond to paramodular forms on  $GSp(2)_{\mathbf{A}_{\mathbf{Q}}}$  of level  $d_B$ .

(3) At the Archimedean place  $\mathcal{L}(f, f')$  generates the quaternionic discrete series with minimal  $K_{\infty}$ -type  $\tau_{\kappa}$ , while  $\mathcal{L}_{GSp(2)}(f, \text{JL}(f'))$  generates the large discrete series with the Blattner parameter  $(\kappa + 1, -1)$ . The latter is due to Przebinda [P, Chap.III, §3]. These two discrete series representations have the same infinitesimal character. That is, the functorial correspondence of these two global theta lifts is compatible with the Archimedean local functorial correspondence established by Langlands [Lg-2].

## 2.8 Strictly positive central L-values

As an application of our main results we show the existence of strictly positive central values for the  $L$ -functions in our concern. In this subsection we fix the quaternion algebra  $B$  and a maximal order  $\mathfrak{O}$  of  $B$  as

$$B = \mathbb{Q} + \mathbb{Q} \cdot i + \mathbb{Q} \cdot j + \mathbb{Q} \cdot k \quad (i^2 = j^2 = -1, ij = -ji = k),$$

$$\mathfrak{O} = \mathbb{Z} \cdot \frac{1+i+j+k}{2} + \mathbb{Z} \cdot i + \mathbb{Z} \cdot j + \mathbb{Z} \cdot k.$$

We note that  $d_B = 2$  for this  $B$ . In [M-N-2, §14] we have shown the following:

**Proposition 2.14.** *Suppose that  $\xi = i/2$  and  $\chi$  is unramified at every finite prime, i.e.  $i_p(\chi) = 0$  for any finite prime  $p$ . Let  $D \in \{1, 2\}$  and  $\kappa \geq 12$  (resp.  $\kappa \geq 8$ ) be divisible by 4 (resp. by 8) when  $D = 1$  (resp.  $D = 2$ ). Then there exist Hecke eigenforms  $(f, f') \in S_{\kappa}(D) \times \mathcal{A}_{\kappa}$  such that  $P_{\chi}(f, \gamma_0)P_{\chi}(f', \gamma'_0) \neq 0$  (which implies  $\mathcal{L}(f, f')_{\xi}^{\chi} \neq 0$ ).*

We now assume that  $D = 1$  and  $\chi$  is unramified at every finite prime. From Proposition 2.4, Proposition 2.5 and this proposition we deduce the following:

**Theorem 2.15.** *Under the assumption in Proposition 2.14 there exist Hecke eigenforms  $(f, f') \in S_{\kappa}(1) \times \mathcal{A}_{\kappa}$  such that*

$$L(\Pi, \chi^{-1}, \frac{1}{2}) > 0, \quad L(\Pi', \chi^{-1}, \frac{1}{2}) > 0.$$

The point of our proof for this theorem is to show the positivity of the constants  $C(f, \chi)$  and  $C(f', \chi)$  for  $(f, f')$  in Proposition 2.14. As an immediate consequence of this theorem and Proposition 2.11 we have obtained the theorem as follows:

**Theorem 2.16.** *Keeping the assumption in Proposition 2.14, there are Hecke eigenforms  $(f, f') \in S_\kappa(1) \times \mathcal{A}_\kappa$  such that*

$$L(\mathcal{L}(f, f'), \chi^{-1}, \frac{1}{2}) = L(\mathcal{L}_{\mathrm{GSp}(2)}(f, \mathrm{JL}(f')), \chi^{-1}, \frac{1}{2}) > 0.$$

**Remark 2.17.** We remark that there are several results on the non-negativity of the central-values of the  $L$ -functions in Theorem 2.15 and Theorem 2.16 (cf. [Gu], [J-C], [Lp] etc.).

### 3 Outline of the proof of Proposition 2.5

In order to verify Theorem 2.6 and Theorem 2.9 it remains to show Proposition 2.5. In this section we choose the measure of  $\mathbb{A}_\mathbb{Q}^\times$  so that

$$\mathrm{vol}(\mathbb{Z}_p^\times) = 1 \text{ for any } p < \infty.$$

For the proof of the proposition we need two formulas. The first formula is due to Waldspurger [Wa, Proposition 7]. In order to review it we need several remarks. We first remark that every  $\pi_v''$  is unitary and thus equipped with a unitary inner product. When  $v = p \nmid d_B$  we embed  $E_p^\times$  into  $B_p^\times = \mathrm{GL}_2(\mathbb{Q}_p)$  by  $\iota_\xi$  (cf. §1.4). Recall that, for the quadratic character  $\eta$  attached to the quadratic extension  $E/\mathbb{Q}$ , we have let  $L(\eta, s)$  be the  $L$ -function defined by  $\eta$  and  $L_v(\eta_v, s)$  the local factor of  $L(\eta, s)$  at a place  $v$  (cf. §2.3).

**Proposition 3.1** (Waldspurger). *For  $b = (b_v)_{v \leq \infty} \in B_{\mathbb{A}_\mathbb{Q}}^\times$ ,*

$$\frac{\|P_\chi(f'; b)\|^2}{\langle f', f' \rangle} = \frac{\sqrt{|d_\xi|}}{8\pi} \frac{\zeta(2)L(\Pi', \chi^{-1}, \frac{1}{2})}{L(\pi(\mathrm{JL}(f')), \mathrm{Ad}, 1)} \prod_{v \leq \infty} \alpha_v(f'_v, \chi_v, b_v),$$

where

$$\alpha_v(f'_v, \chi_v, b_v) := \frac{L_v(\eta_v, 1)L_v(\pi'_v, \mathrm{Ad}, 1)}{\zeta_v(2)L_v(\Pi'_v, \chi_v^{-1}, \frac{1}{2})} \int_{\mathbb{Q}_v^\times \setminus E_v^\times} \frac{\langle \pi_v''(tb_v)f'_v, \pi_v''(b_v)f'_v \rangle_v}{\langle f'_v, f'_v \rangle_v} \chi_v(t)^{-1} dt$$

with an inner product  $\langle *, * \rangle_v$  of  $\pi_v''$  for each place  $v$ .

Here we note that the normalization of our measure  $dt$  differs from that of [Wa] by  $\frac{\sqrt{|d_\xi|}}{4\pi}$ -multiple, and furthermore note that the toral integral in the sense of [Wa] is replaced by ours under the normalization of the measure of  $\mathbb{A}_\mathbb{Q}^\times$  mentioned above.

The second formula is the well-known explicit formula for zonal spherical functions by Macdonald [Ma, Chap.V, §3, (3.4)]. We put

$$\phi_v(g) := \frac{\langle \pi_v''(g)f'_v, f'_v \rangle_v}{\langle f'_v, f'_v \rangle_v} \quad (g \in B_v^\times)$$

for each place  $v$ . We note that  $\pi_p''$  at  $p \nmid d_B$ , which is isomorphic to  $\pi_p'$ , is a unitary unramified principal series representation of  $\mathrm{GL}_2(\mathbb{Q}_p)$  with a spherical vector  $f'_p$ . The Satake parameter of  $\pi_p''$  is of the form  $(\alpha_p, \alpha_p^{-1}) \in (\mathbb{C}^\times)^2$  with  $|\alpha_p| = 1$ . We see that  $\phi_p(g)$  is a zonal spherical function on  $\mathrm{GL}_2(\mathbb{Q}_p)$  for  $v = p \nmid d_B$ .

**Proposition 3.2** (Macdonald). *Let  $p$  be a finite prime not dividing  $d_B$ . For  $a_m := \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}$  with  $m \geq 0$  we have*

$$\phi_p(a_m) = \frac{p^{-\frac{m}{2}}}{1+p^{-1}} \left( \alpha_p^m \frac{1-p^{-1}\alpha_p^{-2}}{1-\alpha_p^{-2}} + \alpha_p^{-m} \frac{1-p^{-1}\alpha_p^2}{1-\alpha_p^2} \right).$$

This formula is useful to evaluate the local integrals involved in  $\alpha_v(f'_v, \chi_v, b_v)$  for  $v = p$  not dividing  $d_B$ . The evaluation of such local integrals at other places is settled by a direct calculation. We furthermore note that  $\alpha_v(f'_v, \chi_v, b_v)$ 's have factors contributed by ratios of local  $L$ -functions, which are computed by Lemma 2.1, Lemma 2.2 and Lemma 2.3. We now state the following formula.

**Proposition 3.3.** (1) *For  $p \nmid d_B$ ,*

$$\alpha_p(f'_p, \chi_p, \gamma'_{0,p}) = \begin{cases} 1 & (i_p(\chi) = 0), \\ p^{-i_p(\chi)} L_p(\eta_p, 1)^2 & (i_p(\chi) > 0). \end{cases}$$

(2) *When  $p \mid d_B$  we have*

$$\alpha_p(f'_p, \chi_p, \gamma'_{0,p}) = \begin{cases} r_p(1-p^{-1}) & (p \text{ is inert or } p \text{ is ramified and } \omega_p = \delta_P), \\ 0 & (p \text{ is ramified and } \omega_p \neq \delta_P), \end{cases}$$

where  $r_p$  is the ramification index of  $p$  for the quadratic extension  $E/\mathbb{Q}$  (cf. §2.4).

(3) *When  $v = \infty$  we have*

$$\alpha_\infty(f'_\infty, \chi_\infty^{-1}, \gamma'_{0,\infty}) = \frac{\kappa+1}{4} \frac{\langle f'_{\infty,\kappa}, f'_{\infty,\kappa} \rangle_\infty}{\langle f'_\infty, f'_\infty \rangle_\infty},$$

where see §2.4 for  $f'_{\infty,\kappa}$ .

As a result of Proposition 3.1 and Proposition 3.3 we have proved Proposition 2.5.

## References

- [B] S. Böcherer, Bemerkungen über die Dirichletreihen von Koecher und Maass, Math. Gottingensis, Schriftenr. d. Sonderforschungsbereichs Geom. Anal. 68, (1986).
- [E-1] M. Eichler, Über die Darstellbarkeit von Modulformen durch Thetareihen, J. Reine Angew. Math., 195, (1955) 156-171.
- [E-2] M. Eichler, Quadratische Formen und Modulfunktionen, Acta Arith., 4 (1958) 217-239.
- [F-M] M. Furusawa and K. Martin, On central critical values of the degree four L-functions for  $\mathrm{GSp}(4)$ : the fundamental lemma II, to appear in Amer. J. Math.

- [F-S] M. Furusawa and J. Shalika, On central critical values of the degree four  $L$ -functions for  $GSp(4)$ : The fundamental lemma, Mem. Amer. Math. Soc., vol.164, No.782 (2003).
- [Gr-W] B. Gross and N. Wallach, On quaternionic discrete series representations, and their continuations, J. Reine. Angew. Math., 481 (1996), 73-123.
- [Gu] J. Guo, On the positivity of the central critical values of automorphic  $L$ -functions for  $GL(2)$ , Duke Math. J., 83 (1996) 157-190.
- [H-K] M. Harris and S. Kudla, Arithmetic automorphic forms for the non-holomorphic discrete series of  $GSp(2)$ , Duke Math. J., 66 (1992) 59-121.
- [I] T. Ibukiyama, Paramodular forms and compact twist, Automorphic forms on  $GSp(4)$ , Proceedings of the 9th Autumn workshop on number theory, (2006) 37-48.
- [J] H. Jacquet, Automorphic forms on  $GL(2)$ , Part II, Lecture Notes in Math., 278, Springer-Verlag, (1972).
- [J-C] H. Jacquet and Nan Chen, Positivity of quadratic base change  $L$ -functions, Bull. Soc. Math. France, 129 (2001) 33-90.
- [J-L] H. Jacquet and R. Langlands, Automorphic forms on  $GL(2)$ , Lecture Notes in Math., 114, Springer-Verlag, (1970).
- [Lg-1] R. Langlands, Problems in the theory of automorphic forms, Lecture Notes in Math., 170, Springer-Verlag, (1970) 18-86.
- [Lg-2] R. Langlands, On the classification of irreducible representations of real algebraic groups, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., 31 Amer. Math. Soc., Providence, RI (1989), 101-170.
- [Lp] E. Lapid, On the nonnegativity of Rankin-Selberg  $L$ -functions at the center of symmetry, Internat. Math. Res. Notices, (2003) no.2, 65-75.
- [Ma] I. Macdonald, Spherical functions and Hall polynomials, Oxford University Press, (1979).
- [Mi] T. Miyake, Modular forms, Springer-Verlag, (1989).
- [Mu] A. Murase, CM-values and central  $L$ -values of elliptic modular forms (II), Preprints of MPIM, MPIM2008-30.
- [M-N-1] A. Murase and H. Narita, Commutation relations of Hecke operators for Arakawa lifting, Tohoku Math. J., 60 (2008) 227-251.
- [M-N-2] A. Murase and H. Narita, Fourier expansion of Arakawa lifting I: An explicit formula and examples of non-vanishing lifts, to appear in Israel Journal of Mathematics.
- [Na] H. Narita, Fourier-Jacobi expansion of automorphic forms on  $Sp(1, q)$  generating quaternionic discrete series, J. Funct. Anal., 239 (2006), 638-682.



- [No] M. Novodvorsky, Automorphic  $L$ -functions for symplectic group  $GS(4)$ , Proc. Symp. Pure Math., 33 (2) (1979), 87-95.
- [O] T. Okazaki, Paramodular forms on  $GS(2, \mathbb{A})$ , preprint.
- [P] T. Przebinda, The oscillator duality correspondence for the pair  $O(2, 2)$  and  $Sp(2, \mathbb{R})$ , Memoirs of A. M. S., vol.70, No.403 (1989).
- [Sh] H. Shimizu, Theta series and automorphic forms on  $GL_2$ , J. Math. Soc. Japan, 24 (1972) 638-683.
- [Su] T. Sugano, On holomorphic cusp forms on quaternion unitary groups of degree 2, J. Fac. Sci. Univ. Tokyo, 31 (1985) 521-568.
- [Wa] J. Waldspurger, Sur les valeurs de certaines fonctions  $L$  automorphes en leur centre de symétrie, Compos. Math., 54 (1985) 173-242.

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